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The behaviour of clusters of spheres falling in a viscous fluid

Part 2. Slow motion theory

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A theoretical study is made of the behaviour of clusters of spheres falling in a viscous fluid under the assumptions that: (*a*) inertial effects are negligible, (*b*) the distance between any two spheres is large compared with their radii. The equations of motion are derived and solved for a number of particular cases and the results compared with the experimental observations of the same motions reported in the preceding paper (Jayaweera, Mason & Slack 1964). For three or four spheres, initially in a horizontal line, the theory is in general agreement with the experiments. Three spheres forming an isosceles triangle are shown to oscillate about the horizontal and about the equilateral shape, so that this theory is unable to explain the observed tendency for three to six spheres to form a regular horizontal polygon. The stability of the steady configuration of n spheres at the vertices of a regular horizontal polygon is examined and it is found that the configuration is only stable for $n < 7$, which explains why this configuration is not observed for more than six spheres.

1. The equations of motion for a cluster of falling spheres

The preceding paper by Jayaweera, Mason & Slack (1964), hereafter referred to as JMS, reports some observations made of the relative motions of a cluster of spheres falling in a viscous fluid at low Reynolds numbers. The present paper contains a theoretical investigation of the small Reynolds number phenomena in an attempt to answer the question: How many of the results reported in JMS can be explained by use of the Stokes slow-motion equations? In addition to the neglect of the inertial terms in the Navier–Stokes equations, the further simplifying assumption is made that the distances between the spheres are large compared with their radii. The use of this restriction retains the most important terms in the mutual interaction of the spheres and enables the fluid motion to be found by the superposition of the motions produced by each sphere in the absence of the others. It rules out, however, any consideration of the motion when two or more spheres are nearly in contact. Since the Stokes equations are only valid for small Reynolds numbers and for the region near the spheres, the use of these assumptions will be justified if $Re \ll 1$, $a/s \ll 1$ and $Re\,(s/a) \ll 1$, where Re is the Reynolds number (based on the diameter of the sphere, its speed of free fall in the viscous fluid and the kinematic viscosity of the fluid), a the radius of a sphere and s the distance between the spheres. Since the Reynolds number is small and the spheres and fluid are of comparable density, the time taken for any sphere to adjust its velocity in response to changes in the fluid velocity can be neglected. This implies that the hydrodynamical forces on each sphere always balance its weight in the fluid.

The familiar Stokes solution for a sphere of radius a moving with speed U in fluid at rest gives a fluid velocity $3aU/2r$ at a point distant r from the sphere in the line of motion and a velocity $3aU/4r$ at a point in the plane perpendicular to the line of motion, where terms $O(a^3/r^3)$ have been neglected. It follows that, to this order of approximation, the forces on two spheres moving with velocities U_1 and U_2 along their line of centres are $6\pi a\mu(U_1 - 3aU_2/2r)$ and $6\pi a\mu(U_2 - 3aU_1/2r)$, and if the velocities are perpendicular to the line of centres the forces are $6\pi a\mu(U_1 - 3aU_2/4r)$ and $6\pi a\mu(U_2 - 3aU_1/4r)$, in both cases in the opposite direction to the velocities. By combining these results, we see that the force on a sphere moving with velocity $\mathbf{v}_1$ in the presence of another sphere moving with velocity $\mathbf{v}_2$ and with position vector $\mathbf{r}$ relative to the first sphere is

$$-6\pi a\mu\left\{\mathbf{v}_1 - \frac{3a}{2r^3}(\mathbf{v}_2 . \mathbf{r})\,\mathbf{r} - \frac{3a}{4r}\left(\mathbf{v}_2 - \frac{\mathbf{v}_2 . \mathbf{r}}{r^2}\mathbf{r}\right)\right\}. \tag{1}$$

It is convenient to use the radius a of each sphere as the unit of length and the terminal velocity $2\sigma a^2 g/9\mu$ as the unit of velocity, where σ is the difference in density of the sphere and the fluid and μ the fluid viscosity. If the positions and velocities of the spheres are denoted by $\mathbf{r}_i$, $\mathbf{v}_i$ $(i = 1, 2, \ldots, n)$, the non-dimensional equations expressing the balance of the forces on each sphere are

$$\mathbf{v}_i - \sum_{j\neq i}\frac{3}{4r_{ij}}\mathbf{v}_j - \sum_{j\neq i}\frac{3}{4r_{ij}^3}(\mathbf{v}_j . \mathbf{r}_{ij})\,\mathbf{r}_{ij} = \mathbf{z}, \tag{2}$$

where $\mathbf{z}$ is a unit vector in the downward vertical, $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$ and terms $O(1/r_{ij}^3)$ have been neglected. The solution of these equations, neglecting terms $O(1/r_{ij}^2)$, is

$$\mathbf{v}_i = \mathbf{z} + \sum_{j\neq i}\frac{3}{4r_{ij}}\mathbf{z} + \sum_{j\neq i}\frac{3}{4r_{ij}^3}(\mathbf{z} . \mathbf{r}_{ij})\,\mathbf{r}_{ij}. \tag{3}$$

If $n = 2$, the velocities of both spheres are the same and there is no change in configuration. The spheres move in a direction lying between the direction of their line of centres and the downward vertical. The reason for the horizontal drift of the spheres is the greater resistance offered by the fluid to the spheres when they are moving abreast of each other than when they are in line. For $n > 2$, the velocities are not in general all equal, and the configuration will change. It is clear, however, that, if the spheres are at the vertices of a regular horizontal polygon, the configuration is steady and, at least for small values of n, this is the only possible steady configuration.

The equations determining the paths of the spheres, and hence the change in configuration, are $d\mathbf{r}_i/dt = \mathbf{v}_i$, with $\mathbf{v}_i$ given by (3) or, changing the time-scale by a factor of $\frac{4}{3}$,

$$\frac{d\mathbf{r}_i}{dt} = \tfrac{4}{3}\mathbf{z} + \sum_{j\neq i}\frac{1}{r_{ij}}\mathbf{z} + \sum_{j\neq i}\frac{\mathbf{z} . \mathbf{r}_{ij}}{r_{ij}^3}\mathbf{r}_{ij}. \tag{4}$$

The relative motions of the spheres are unaffected by the term $\frac{4}{3}\mathbf{z}$. An alteration of scale in the configuration of the spheres is equivalent to a change in the time-scale, since the remaining terms in (4) are homogeneous functions of the positions. Without any loss of generality, any convenient length in the initial configuration can be chosen as unit.

With $n = 3$ the relative motion is given by the two vector equations

$$\frac{d\mathbf{r}_{12}}{dt} = \left(\frac{1}{r_{13}}\mathbf{z} + \frac{\mathbf{z}\,.\,\mathbf{r}_{13}}{r_{13}^3}\mathbf{r}_{13}\right) - \left(\frac{1}{r_{23}}\mathbf{z} + \frac{\mathbf{z}\,.\,\mathbf{r}_{23}}{r_{23}^3}\mathbf{r}_{23}\right), \tag{5}$$

$$\frac{d\mathbf{r}_{23}}{dt} = \left(\frac{1}{r_{21}}\mathbf{z} + \frac{\mathbf{z}\,.\,\mathbf{r}_{21}}{r_{21}^3}\mathbf{r}_{21}\right) - \left(\frac{1}{r_{31}}\mathbf{z} + \frac{\mathbf{z}\,.\,\mathbf{r}_{31}}{r_{31}^3}\mathbf{r}_{31}\right). \tag{6}$$

A simple result follows immediately from these equations. If $\mathbf{\Delta}$ is the vector area of the triangle formed by the three spheres, defined by $2\mathbf{\Delta} = \mathbf{r}_{12} \times \mathbf{r}_{32}$, the rate of change of $\mathbf{\Delta}$ is

$$\frac{d\mathbf{\Delta}}{dt} = \mathbf{z} \times \frac{1}{2}\left(\frac{\mathbf{r}_{12}}{r_{12}} + \frac{\mathbf{r}_{23}}{r_{23}} + \frac{\mathbf{r}_{31}}{r_{31}}\right).$$

Hence $\mathbf{z}\,.\,d\mathbf{\Delta}/dt = 0$; i.e. the horizontal projection of the triangle is of constant area. No other simple result has been found, so that even for $n = 3$ it is necessary to choose a configuration displaying some symmetry to reduce the complexity of the problem. Particular examples which form part of the experimental evidence will be discussed in the following sections.

2. Three spheres falling in a vertical plane

Three spheres initially placed in a horizontal line will always lie in the same vertical plane. If x is measured in the horizontal direction in the plane of motion and z vertically downwards, the co-ordinates of two of the spheres relative to the third may be written (x_1, z_1) and (x_2, z_2) and the equations (5) and (6) reduce to

$$\left.\begin{aligned}
\frac{dz_1}{dt} &= \frac{2(z_1-z_2)^2+(x_1-x_2)^2}{r_3^3} - \frac{2z_2^2+x_2^2}{r_2^3},\\
\frac{dz_2}{dt} &= \frac{2(z_1-z_2)^2+(x_1-x_2)^2}{r_3^3} - \frac{2z_1^2+x_1^2}{r_1^3},\\
\frac{dx_1}{dt} &= \frac{(z_1-z_2)(x_1-x_2)}{r_3^3} - \frac{z_2x_2}{r_2^3},\\
\frac{dx_2}{dt} &= \frac{(z_1-z_2)(x_1-x_2)}{r_3^3} - \frac{z_1x_1}{r_1^3},
\end{aligned}\right\} \tag{7}$$

where $\quad r_1^2 = z_1^2+x_1^2, \quad r_2^2 = z_2^2+x_2^2, \quad r_3^2 = (z_1-z_2)^2+(x_1-x_2)^2.$

The initial conditions are $z_1 = z_2 = 0$, $x_1 = 1, x_2 = -c$, so that the motion is given relative to the initially central sphere and all cases are covered by $1 \leqslant c < \infty$. The equations (7) were solved numerically for a number of values of c and some of the results are shown in figure 1. The calculations show that one sphere, initially the one further from the centre, lags behind the others, moves towards the centre and falls between them, the process continuing, with another sphere lagging behind, for a certain number of times, until two of the spheres are relatively close together, when they fall together as a pair, leaving the third sphere behind. In all the calculated cases, one of the spheres was eventually left behind, but the particular one that was separated was very sensitive to variations in c.

These results are in general agreement with the observations described in JMS §4 (*a*). The calculations showed that the particular sphere that is left behind is the one initially further from the centre for $c = 3$ and 4, the one initially nearer the centre for $c = 1{\cdot}1$, $1{\cdot}2$ and 2 and the central sphere for $c = 1{\cdot}5$. Thus the theory and the experiments agree except for the central range $1{\cdot}2 < c < 2$, which is the range where the behaviour is most sensitive to changes in c.

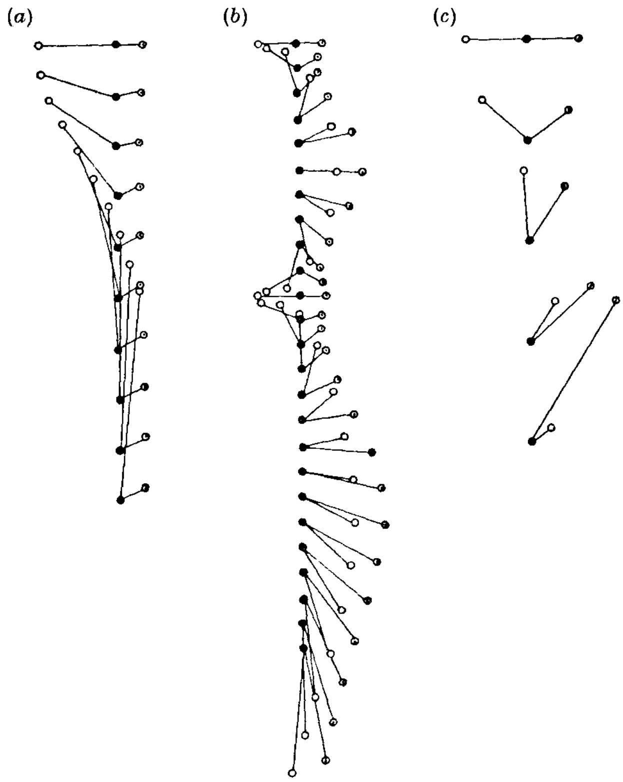

FIGURE 1. Positions of three spheres, initially in a horizontal line, relative to the central sphere. (*a*) $c = 3$; (*b*) $c = 1{\cdot}5$; (*c*) $c = 1{\cdot}2$.

3. Four spheres falling in a vertical plane

Four spheres initially placed symmetrically in a horizontal line will fall in a vertical plane and retain their symmetry. With the same axes as in the previous section, the co-ordinates of the spheres can be written (x_1, z), $(x_2, 0)$, $(-x_1, z)$, $(-x_2, 0)$ and the equations (4) give

$$\frac{dx_1}{dt} = z\left\{\frac{x_2+x_1}{((x_2+x_1)^2+z^2)^{\frac{3}{2}}} - \frac{x_2-x_1}{((x_2-x_1)^2+z^2)^{\frac{3}{2}}}\right\},$$

$$\frac{dx_2}{dt} = -z\left\{\frac{x_2+x_1}{((x_2+x_1)^2+z^2)^{\frac{3}{2}}} + \frac{x_2-x_1}{((x_2-x_1)^2+z^2)^{\frac{3}{2}}}\right\},$$

$$\frac{dz}{dt} = \frac{1}{2x_1} - \frac{1}{2x_2}.$$

Writing $x_2+x_1 = x$, $x_2-x_1 = y$, and absorbing a factor 2 into the time-scale, we have the simpler equations

$$\frac{dx}{dt} = \frac{-yz}{(y^2+z^2)^{\frac{3}{2}}}, \tag{8}$$

$$\frac{dy}{dt} = \frac{-xz}{(x^2+z^2)^{\frac{3}{2}}}, \tag{9}$$

$$\frac{dz}{dt} = \frac{y}{x^2-y^2}. \tag{10}$$

The initial conditions are $z = 0$, $x = 1$, $y = c$, $0 < c < 1$. The initial distance between the two outer spheres is $1+c$ and between the two inner spheres $1-c$.

The possible nature of the solution of these equations can be discovered by the following arguments. If $x = \pm y$ at any instant, $x = \pm y$ for all time from (8) and (9). Hence $x \pm y$ are of constant sign; i.e. they are both positive. Also, x and y are even functions and z is an odd function of t. Suppose $z = 0$ at some time $T > 0$, at which time $x = x_1$, $y = y_1$. Then at time $t = -T$, $z = 0$, $x = x_1$, $y = y_1$ and the solution is periodic with period $2T$. In this type of motion, the two pairs of spheres on either side of the central line oscillate in the horizontal direction as they fall, the members of each pair rotating round each other. If, however, z never vanishes for $t > 0$, $z > 0$ and y is a monotonic decreasing function. If $dz/dt = 0$ for any t then, for larger t, dz/dt and d^2z/dt^2 are both negative and z must vanish. Hence, dz/dt never vanishes and so $y > 0$. Since y is a monotonic decreasing function y tends to a limit, α, say, where $0 \leqslant \alpha < c$ and x tends to a limit β, where $\alpha \leqslant \beta < 1$, and z tends to infinity. In this type of motion, the inner pair of spheres fall together, leaving the outer pair behind, the outer pair remaining further apart than the inner pair and so falling more slowly. The equations (8)–(10) were solved numerically for a range of values of c. For $0 < c \leqslant 0{\cdot}65$, the periodic type of motion occurs, with a period increasing very rapidly as c increases from 0·5 to 0·65. It is not possible to prove by numerical integration that the periodic motion does *not* occur for larger values of c. All that can be proved is that the period must be longer than the range of t for which the numerical solution is obtained. The solution for $c = 0{\cdot}7$ showed that the period, if it exists, must be greater than 100, compared with a period of 10·75 for $c = 0{\cdot}65$ and 5·1 for $c = 0{\cdot}6$. The results suggest, but do not conclusively prove, that, for $0{\cdot}7 < c < 1$, the second type of motion occurs in which the two inner spheres separate from the two outer ones. It is not surprising that this type of motion should occur for values of c close to 1, since then the two inner spheres are very close together. It is in fact possible to prove that, for c sufficiently large, the separating type of motion must occur. The equations (8)–(10) can be written

$$\frac{dX}{dZ} = -\frac{X^{\frac{1}{2}}(X-Y)}{(Y+Z)^{\frac{3}{2}}}, \quad \frac{dY}{dZ} = -\frac{X^{\frac{1}{2}}(X-Y)}{(X+Z)^{\frac{3}{2}}}, \tag{11}$$

where $X = x^2$, $Y = y^2$, $Z = z^2$ and the initial value of Y is $1-\epsilon$. X, Y and $X-Y$ are all monotonically decreasing functions of Z. If sequences of functions X_n, Y_n are defined by the relations

$$\left.\begin{aligned} \frac{dX_{n+1}}{dZ} = -\frac{X_n^{\frac{1}{2}}(X_n-Y_n)}{(Y_n+Z)^{\frac{3}{2}}}, \quad \frac{dY_{n+1}}{dZ} = -\frac{X_n^{\frac{1}{2}}(X_n-Y_n)}{(X_n+Z)^{\frac{3}{2}}}, \\ X_{n+1} = 1, \quad Y_{n+1} = 1-\epsilon, \quad \text{at} \quad Z = 0, \end{aligned}\right\} \tag{12}$$

X_n and Y_n tend to the solutions X and Y of (11). If $1 \geqslant X_n > A_n$, $1-\epsilon \geqslant Y_n > B_n$, the relations (12) can be used to show that $1 \geqslant X_{n+1} > A_{n+1}$, $1-\epsilon \geqslant Y_{n+1} > B_{n+1}$, where

$$A_{n+1} = 1-2\epsilon/B_n^{\frac{1}{2}}, \quad B_{n+1} = 1-\epsilon-2\epsilon/A_n^{\frac{1}{2}}.$$

The sequence B_n can be shown to tend to a non-zero limit if $\epsilon < 0{\cdot}112$. In terms of the original variables, y is always positive and the separating type of motion occurs for $c > 0{\cdot}94$. (This does not imply that the periodic motion necessarily occurs for $c < 0{\cdot}94$.)

The experimental evidence of JMS §4(a) supports the periodic motion. Because of the finite depth of the tank, the separating motion would not be distinguishable from a periodic motion with long period.

4. Three spheres forming a horizontal triangle

The most important features of the motion of three spheres when they do not lie in a vertical plane can be illustrated by the particular case when they form an isosceles triangle with the unequal side horizontal. The symmetry of this configuration remains throughout the motion, which results in a considerable reduction in the complexity of the motion. If x and y are horizontal co-ordinates with the y-axis parallel to the base of the triangle, the positions of the two base spheres relative to the sphere at the apex of the triangle are $\mathbf{r}_{13} = (x, y, z)$, $\mathbf{r}_{23} = (x, -y, z)$. The equations of motion (5) and (6) then reduce to the three equations

$$\frac{dx}{dt} = -\frac{zx}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \tag{13}$$

$$\frac{dy}{dt} = \frac{zy}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \tag{14}$$

$$\frac{dz}{dt} = \frac{1}{2y} - \frac{x^2+y^2+2z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}. \tag{15}$$

It follows at once that xy is constant, which is an expression of the result previously obtained about the horizontal projection of the area of the triangle, and without loss of generality we may take $xy = 1$. Writing p for the cosine of the base angle of the triangle, we have

$$\frac{1}{p^2} = \frac{x^2+y^2+z^2}{y^2} = 1+\frac{1}{y^4}+\frac{z^2}{y^2}. \tag{16}$$

Differentiating this result and using the value of dz/dy obtained from (14) and (15), we can obtain the single separable equation

$$\frac{dp}{dy} + \frac{1-6p+4p^3}{2y} = 0, \tag{17}$$

which has the solution

$$y = \frac{A|p-0{\cdot}17|^{0{\cdot}35}}{(1{\cdot}13-p)^{0{\cdot}21}(1{\cdot}30+p)^{0{\cdot}14}}, \tag{18}$$

where A is a constant and the numerical terms are correct to two decimal places. Since, from (16), $1/p^2 \geqslant 1+1/y^4$, only part of the range of values given by (18) represents a possible shape for the triangle. The curves representing equation (18) and the equation

$$y^4 = p^2/(1-p^2) \tag{19}$$

are sketched in figure 2. The triangle is determined completely by a point P on the curve, except for the sign of z, and there are two possible ranges, between A and B and between C and D. At points where the two curves cross, $z = 0$ and the direction of motion along the curve is that of y increasing if z is positive, and the reverse direction if z is negative (see equation (14)). There are

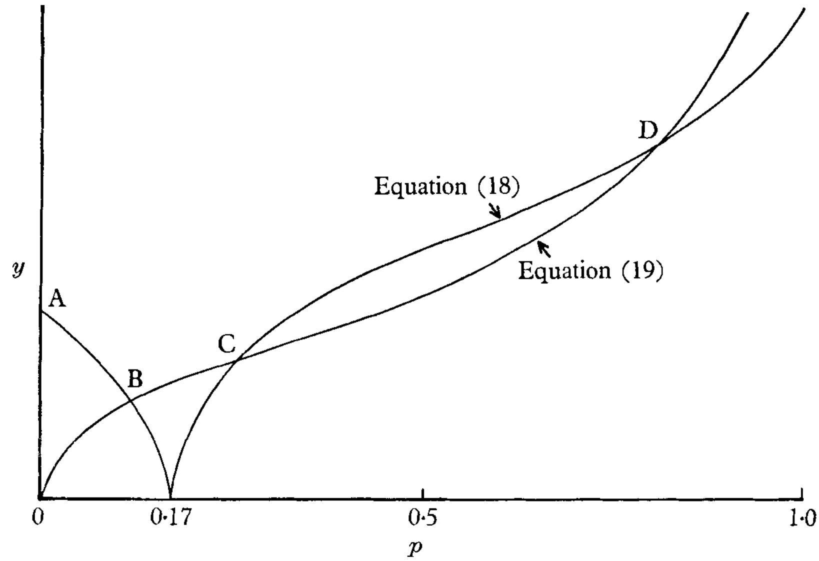

FIGURE 2. Sketch of equations (18) and (19). The parts of the curve (18) lying between A and B and between C and D determine possible shapes for the triangle formed by three spheres.

two cases to be considered. If $p < 0{\cdot}17$ at any time, the point P representing the configuration of the triangle always lies in the arc AB. If z is negative, P moves towards B, where $z = 0$, and then reverses its direction and moves towards A, but never reaches A, since, at A, z is infinite. In terms of the triangle formed by the spheres, this means that, if the apex angle is less than about 20° and the two base spheres are initially above the apex sphere, they fall past the apex sphere, the angle at the apex increasing to a maximum when the plane is horizontal and decreasing again as the two spheres move away from the apex. For such narrow-angled triangles, the coupling between the apex and the base is so weak that the base spheres and apex sphere fall almost independently, only influencing each other when the plane is nearly horizontal. The other case is when $p > 0{\cdot}17$, when P moves backwards and forwards along the arc CD, the plane being horizontal when P is at the two ends of its path, at which points the angles have their extreme values. Starting from C, the cycle of motion is that the base spheres fall below the apex to a maximum distance and return to the horizontal position, the apex angle steadily increasing, and then the apex sphere falls below the base and returns to the horizontal, the apex angle steadily decreasing, until the original position C is regained. It is easy to show, by a consideration of the slopes of the two curves with equations (18) and (19), that the points C and D lie on either side of the point where $p = \frac{1}{2}$, so that the motion of the triangle can be described as an oscillation about the horizontal and about the equilateral shape. The coupling between the spheres is now strong enough to ensure that one cannot be separated from the others.

The observations made of this type of motion are described in JMS §5(v). The variations in shape preserve the horizontal area and the oscillations agree with the theory. The spiral motion of the apex sphere, instead of the predicted plane motion, is probably a consequence of the base line not being exactly horizontal, resulting in a further oscillation about a horizontal axis. The present theory, however, does not explain the decay of the oscillations. The observations suggest the presence of a small damping force, which would eventually destroy the oscillations and leave the spheres in their steady configuration, i.e. a horizontal equilateral triangle. Since the present theory is unable to explain the attaining of the steady configuration in a simple case of three spheres, it is presumably unable to do so for three spheres initially placed arbitrarily or for higher numbers of spheres. On the basis of the theory for the isosceles triangle, it may be conjectured that, in the general case, the spheres will oscillate about the steady configuration, provided they are sufficiently evenly distributed to ensure that one or more spheres do not get left behind by the remainder. The damping factor which is required to explain the achievement of the steady configuration is presumably an inertial effect and so is outside the scope of the present investigation.

5. The stability of the steady configuration

Although the observed attainment of the steady configuration for 3, 4, 5 and 6 spheres cannot be explained by this theory, there remains the question of the different behaviour of more than 6 spheres, which do *not* form a regular polygon. Since 7 spheres placed at the vertices of a regular heptagon do not remain in this steady configuration, it seems likely that the reason for the change in behaviour is connected with the stability of the regular pattern to small disturbances which alter the relative positions of the spheres.

If $\mathbf{x}$ and $\mathbf{y}$ are two perpendicular horizontal unit vectors and an origin is taken at the centre of the polygon, the position vectors of the centres of the n spheres are

$$\mathbf{r}_i = \mathbf{x}\cos(2\pi i/n) + \mathbf{y}\sin(2\pi i/n) \quad (i = 1, 2, \ldots, n).$$

A small disturbance will alter the positions of the spheres to

$$\mathbf{r}_i = \mathbf{x}\cos(2\pi i/n) + \mathbf{y}\sin(2\pi i/n) + x_i\mathbf{x} + y_i\mathbf{y} + z_i\mathbf{z},$$

where x_i, y_i, z_i are small, and it is the behaviour of these $3n$ quantities that will determine the stability or instability of the configuration. The vector $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$ is

$$\mathbf{r}_{ij} = -2\sin\{\pi(i+j)/n\}\sin\{\pi(i-j)/n\}\,\mathbf{x} + 2\cos\{\pi(i+j)/n\}\sin\{\pi(i-j)/n\}\,\mathbf{y} + (x_i - x_j)\,\mathbf{x} + (y_i - y_j)\,\mathbf{y} + (z_i - z_j)\,\mathbf{z},$$

and the magnitude of this vector, which is required in the equations of motion, is

$$r_{ij} = 2\left|\sin\frac{\pi(i-j)}{n}\right|\left\{1-\frac{(x_i-x_j)\sin\{\pi(i+j)/n\}}{2\sin\{\pi(i-j)/n\}}+\frac{(y_i-y_j)\cos\{\pi(i+j)/n\}}{2\sin\{\pi(i-j)/n\}}\right\}, \quad (20)$$

neglecting terms of the second order in x_i, y_i, z_i. To the same order the equations of motion (4) become, on separating the vertical and horizontal components,

$$\frac{dz_i}{dt} = \sum_{j\neq i}\frac{1}{4\sin\{\pi(i-j)/n\}\,|\sin\{\pi(i-j)/n\}|}\left\{(x_i-x_j)\sin\frac{\pi(i+j)}{n}-(y_i-y_j)\cos\frac{\pi(i+j)}{n}\right\},$$

$$\frac{dx_i}{dt}\mathbf{x}+\frac{dy_i}{dt}\mathbf{y} = \sum_{j\neq i}\frac{z_i-z_j}{4\sin\{\pi(i-j)/n\}\,|\sin\{\pi(i-j)/n\}|}\left\{-\sin\frac{\pi(i+j)}{n}\mathbf{x}+\cos\frac{\pi(i+j)}{n}\mathbf{y}\right\}.$$

The range of values for i is 1 to n and the summations are over the same range of values for j, excluding $j = i$. Differentiating the first set of equations, the second derivatives of the z_i can be expressed in terms of the first derivatives of the x_i and the y_i, which in turn can be expressed in terms of the z_i by the second set of equations. Performing this elimination, we obtain

$$16\frac{d^2z_i}{dt^2} = \sum_{j\neq i}\frac{1}{\sin\{\pi(i-j)/n\}|\sin\{\pi(i-j)/n\}|}\left\{\sum_{k\neq i}\frac{-\cos\{\pi(j-k)/n\}(z_i-z_k)}{\sin\{\pi(i-k)/n\}|\sin\{\pi(i-k)/n\}|}\right.$$
$$\left.+\sum_{l\neq j}\frac{\cos\{\pi(i-l)/n\}(z_j-z_l)}{\sin\{\pi(j-l)/n\}|\sin\{\pi(j-l)/n\}|}\right\}. \quad (21)$$

Because of the symmetry of the undisturbed configuration, it is clear that, if the form of the equation (21) for one value of i is known, the rest of the equations can be written down by cyclic interchange of the terms. With $i = n$, and some manipulation of the equation, the equation for z_n becomes

$$4\frac{d^2z_n}{dt^2} = \sum_{j=2}^{n-1}\sum_{k=1}^{j-1}\frac{-\cos\{\pi(j-k)/n\}}{4\sin^2(\pi j/n)\sin^2(\pi k/n)}(2z_n-z_{j-k}-z_{n-j+k}). \quad (22)$$

Writing this equation in the form

$$4(d^2z_n/dt^2) = a_nz_n+a_1z_1+a_2z_2+\ldots+a_{n-1}z_{n-1}, \quad (23)$$

we can at once write down the remaining equations, namely

$$4(d^2z_1/dt^2) = a_nz_1+a_1z_2+\ldots+a_{n-1}z_n,$$
$$4(d^2z_2/dt^2) = a_nz_2+a_1z_3+\ldots+a_{n-1}z_1,$$

and so on. These equations have solutions of the form

$$z_i = Z_i\exp(\pm\tfrac{1}{2}\lambda^{\frac{1}{2}}t) \quad (i = 1, 2, \ldots, n),$$

if λ is a latent root of the matrix

$$\mathbf{A} = \begin{pmatrix} a_n & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_n & a_1 & \cdots & a_{n-2} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_1 & a_2 & a_3 & \cdots & a_n \end{pmatrix}.$$

This type of matrix is called a circulant matrix and the latent roots are known to be

$$\lambda = a_n+\omega a_1+\omega^2a_2+\ldots+\omega^{n-1}a_{n-1},$$

where ω is an nth root of unity. Substituting the values of $a_1, a_2, \ldots, a_n$, from (22) and (23) we have

$$\lambda = \sum_{j=2}^{n-1} \sum_{k=1}^{j-1} \frac{-\cos\{\pi(j-k)/n\}}{4\sin^2(\pi j/n)\sin^2(\pi k/n)}(2-\omega^{j-k}-\omega^{k-j}),$$

and, writing $\omega = \exp(2\pi i r/n)$, the latent roots are

$$\lambda_r = \sum_{j=2}^{n-1} \sum_{k=1}^{j-1} \frac{-\cos\{\pi(j-k)/n\}\sin^2\{\pi r(j-k)/n\}}{\sin^2(\pi j/n)\sin^2(\pi k/n)} \quad (r = 1, 2, \ldots, n). \tag{24}$$

All the latent roots are real and $\lambda_r = \lambda_{n-r}$, which are consequences of the symmetry of $\mathbf{A}$, which in turn depends on the symmetry of the polygon about the x-axis.

The n equations (22) have $2n$ independent solutions, $2n-2$ of which are supplied by $\pm\lambda_1, \ldots, \pm\lambda_{n-1}$ and the remaining two solutions by λ_n, which is zero. These last two solutions are essentially neutral solutions, one corresponding to a vertical displacement of the whole polygon and the other to an expansion or contraction of the polygon without change of shape, which results in a small increase or decrease in the rate of fall of the polygon. The latent roots $\lambda_1, \ldots, \lambda_{n-1}$ are equal in pairs ($\lambda_r = \lambda_{n-r}$) except for $r = \frac{1}{2}n$ when n is even, but an examination of the rank of the matrix $(\mathbf{A} - \lambda\mathbf{I})$ shows that there are n independent latent vectors so no solutions of the form $t\exp(\frac{1}{2}\lambda^{\frac{1}{2}}t)$ need be included.

The polygon will be a stable configuration if all the latent roots are negative, but if one is positive the configuration is unstable. Evaluation of the roots shows that, for $n = 3$, 4, 5 and 6, the configurations are stable, but for $7 \leqslant n \leqslant 12$ there is at least one positive root. The number of positive roots increases as n increases—there are 6 when $n = 12$—and they also increase in magnitude. These facts suggest that there are always positive roots for $n > 6$, but it has not been possible to prove this. Certainly it seems physically unlikely that the unstable configuration can change back into a stable one as the number of spheres increases, particularly if, as seems possible, the change from stability to instability is associated with the increasing ratio of the maximum and minimum distances between the spheres as n increases.

6. Unexplained behaviour

The previous sections have explained some of the observations described in JMS. In addition to the observations made for $Re > 1$ and for spheres nearly in contact, which have been excluded from the discussion, there remain some phenomena which occur for small Reynolds number but which are not explicable by Stokes slow-motion theory. The most striking of these is the attaining of a steady configuration by 3–6 spheres, which was mentioned in §5. Another example is the horizontal separation of the spheres as they fall, which has been observed in nearly all the motions described. Since this separation is very slow, there is no difficulty in regarding it as a small inertial effect. The motion of three and four spheres in a vertical plane is in agreement with the theory for $Re < 0{\cdot}16$ but, for larger Re, the spheres leave the vertical plane. This is presumably because the motion in the vertical plane has become unstable and this also must be an inertial effect. There are also the anomalous motions for very low Reynolds numbers, which tend to maintain the initial configuration. This may be an indication that the fluid used is slightly non-Newtonian in the sense that it is able to support a very small stress, which would prevent the small relative motions of the spheres but allow the cluster to fall as a whole.

The rotation of the spheres has so far been neglected. It is produced by asymmetry of the flow on two hemispheres and is $O(a^4/s^4)$ so it is very small for separated spheres and its neglect is consistent with the approximations made in the theory. Its presence in the two-sphere problem and in the motion of the regular polygon of spheres is in accordance with Stokes slow-motion theory.

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REFERENCE

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